Quantum Computing based on Tensor Products Entanglement and EPR Paradox

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Dodecahedra basics

Ddodecahedron adjacency graph

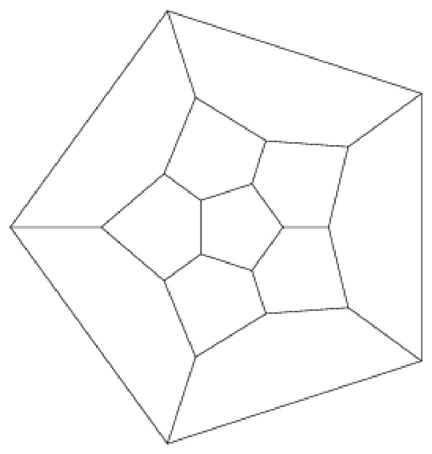
A *dodecahedron* has $n_v = 20$ vertexes, $n_e = 30$ edges and $n_f = 12$ faces. Indeed, since Euclidean sphere has *Euler characteristic* 2:

$$2 = n_v - n_e + n_f.$$

Let $G_D = (V, E)$ be the *dodecahedron adjacency* graph.

- G_D is a *regular* graph, in which each vertex has degree 3.
- Distance between two points: Number of edges in the shortest path connecting those points. Then each vertex has
    - 1 node at distance 0,
    - 3 nodes at distance 1,
    - 6 nodes at distance 2,
    - 6 nodes at distance 3,
    - 3 nodes at distance 4,
    - 1 node at distance 5.

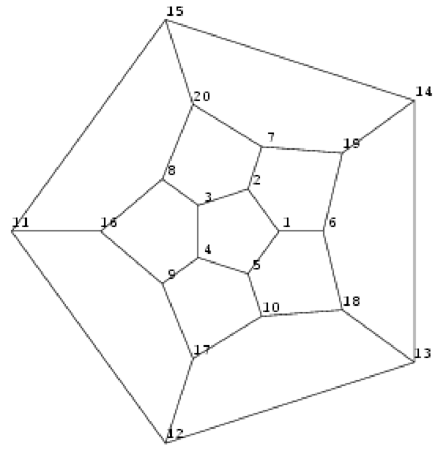
$G_D = (V, E)$



$\forall v \in V$:

- Opposite vertex of v: $\overline{v} \in V$ be the unique vertex at distance 5 to v.
- Set of neighbors of v: $N(v) = \{w \in V | \{v, w\} \in E\}$. It has three elements.
- Next-to-adjacent pairs of v: $M(v) = \{\{v_1, v_2\} \in V^{(2)} | \exists w \in N(v) : \ \{v, v_1, v_2\} = N(w)\}$.

Enumeration of the dodecahedron adjacency graph: $\overline{v_j} = v_k \iff k = 1 + (j + 10) \bmod 20$



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The Game

Alice lives Here and Bob lives FarAway (thousands light years from here). They receive two identical dodecahedra from QT, each having a button at each vertex, as well as precise instructions to align the dodecahedra perfectly parallel.

When they push buttons, either nothing happens or a bell rings and the dodecahedron fires a set fireworks.

The game is the following:

At each step, Alice and Bob select vertexes at their dodecahedra. They do not press the selected buttons. They press the neighbor buttons:

I. If the selected vertexes are opposite then a neighbor button rings if and only if its opposite in other dodecahedron rings.
II. If the selected vertexes are corresponding then one of the six neighbor buttons should ring.

Vertex colorings

How to color

Let us color each vertex with the color white if it rings and with the color black otherwise.

Conditions to succed

A. No pair next-to-adjacent to any vertex can have the same color.

B. No set of the form $N(v) \cup N(\overline{v})$ can be black.

Theorem

No coloring does exist satisfying both conditions **A.** and **B.**

Theorem

State entanglement does allow to build such magic dodecahedra.

R. Penrose. *Shadows of the Mind*. Vintage. London, 1995.

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Self-adjoint operators

An observable in a space $\mathbb{H}_n$ is a self-adjoint linear operator $U : \mathbb{H}_n \to \mathbb{H}_n$, i.e. $U^H = U$.

If $U, V : \mathbb{H}_n \to \mathbb{H}_n$ are observables, $U + V$ is also an observable, but the product UV will be if, for instance, U and V conmute. $UV + VU$ and $i(UV - VU)$ always are observables.

For an observable $U : \mathbb{H}_n \to \mathbb{H}_n$ there exists an ON basis of $\mathbb{H}_n$ consisting of eigenvectors of U. Hence, if $\lambda_0, \dots, \lambda_{k-1}$ are the eigenvalues of U and $L_0, \dots, L_k$ are the corresponding eigenspaces

$$\mathbf{x} \in L_\kappa \implies U(\mathbf{x}) = \lambda_\kappa \mathbf{x}.$$

Consequently, U is represented as

$$U = \sum_{\kappa=0}^{k-1} \lambda_\kappa \pi_{L_\kappa},$$

where for each space $L < \mathbb{H}_n$, $\pi_L : \mathbb{H}_n \to L$ is the orthogonal projection over L.

If $\{\mathbf{v}_0, \dots, \mathbf{v}_{m-1}\}$ is an ON basis of L and $\mathbf{L}$ is the matrix whose columns are these vectors then π_L is represented by $\mathbf{L} \cdot \mathbf{L}^H$.

Since π_{L_κ} is an orthogonal projection, $\forall \mathbf{x} \in \mathbb{H}_n$, $\langle \mathbf{x} - \pi_{L_\kappa}(\mathbf{x}) | \pi_{L_\kappa}(\mathbf{x}) \rangle = 0$, thus

$$\langle \mathbf{x} | \pi_{L_\kappa}(\mathbf{x}) \rangle = \langle \pi_{L_\kappa}(\mathbf{x}) | \pi_{L_\kappa}(\mathbf{x}) \rangle = \| \pi_{L_\kappa}(\mathbf{x}) \|^2.$$

Extended measurement principle

For any observable U, when measuring an n-register $\mathbf{x} \in \mathbb{H}_n$, the output is an eigenvalue λ_κ and the current state will be the normalized projection $\frac{\pi_{L_\kappa}(\mathbf{x})}{\|\pi_{L_\kappa}(\mathbf{x})\|}$. For each eigenvalue λ_κ, the probability that it is the output is[a]

$$\Pr(\lambda_\kappa) = \langle \mathbf{x} | \pi_{L_\kappa}(\mathbf{x}) \rangle. \qquad (1)$$

[a] evidently, $\sum_{\kappa=0}^{k-1} \Pr(\lambda_\kappa) = \sum_{\kappa=0}^{k-1} \|\pi_{L_\kappa}(\mathbf{x})\|^2 = \|\mathbf{x}\|^2 = 1$.

Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2}$$

For any real vector $\mathbf{v} \in \mathbb{R}^3$, let

$$V_{\mathbf{v}} = v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3 = \begin{pmatrix} v_3 & v_1 - i\,v_2 \\ v_1 + i\,v_2 & -v_3 \end{pmatrix}. \tag{3}$$

Whenever **v** is an unit vector, $V_{\mathbf{v}}$ is an observable and it is called the measurement of spin along vector **v**. The eigenvalues of $V_{\mathbf{v}}$ are $-\|\mathbf{v}\|_2, \|\mathbf{v}\|_2$, i.e. they are $-1, 1$ with corresponding eigenvectors

$$\mathbf{u}_{\mathbf{v}0} = \begin{bmatrix} v_3 - \|\mathbf{v}\|_2 \\ v_1 + i\, v_2 \end{bmatrix} = \begin{bmatrix} v_3 - 1 \\ v_1 + i\, v_2 \end{bmatrix} \;,\; \mathbf{u}_{\mathbf{v}1} = \begin{bmatrix} v_3 + \|\mathbf{v}\|_2 \\ v_1 + i\, v_2 \end{bmatrix} = \begin{bmatrix} v_3 + 1 \\ v_1 + i\, v_2 \end{bmatrix}.$$

For any $\mathbf{x} = [x_0 \;\; x_1]^T \in \mathbb{H}_1$ we have

$$\langle \mathbf{x} | V_{\mathbf{v}} \mathbf{x} \rangle = (2x_0 x_1)\, v_1 + \left(x_0^2 - x_1^2\right) v_3$$

thus the expectation of $V_{\mathbf{v}}$ at state **x** is a rotation depending on **x**.

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In 1964 John Bell showed that no physical theory "realistic" and "local", with well defined notions of those terms, can explain all statistical implications of Quantum Mechanics. Thus, any quantum state is "incomplete" when predicting all its physical attributes.

A thought experiment

Let us assume

- Charly prepares two particles.
- He gives one to Alice and the other to Bob.
- Alice is able to perform measurements on her particle about two properties, W and X, to obtain ± 1 values P_W or P_X. She selects randomly W or X.
- Bob as well is able to perform measurements of two properties, Y and Z, to obtain ± 1 values P_Y or P_Z. He selects also randomly Y or Z.
- Let

$$F = WY + XY + XZ - WZ. \tag{4}$$

We have $F = \pm 2$, and half of the combinations give the negative value:

W	X	Y	Z
−1	−1	1	−1
−1	−1	1	1
−1	1	−1	−1
−1	1	1	−1
1	−1	−1	1
1	−1	1	1
1	1	−1	−1
1	1	−1	1

$F = -2$

W	X	Y	Z
−1	−1	−1	−1
−1	−1	−1	1
−1	1	−1	1
−1	1	1	1
1	−1	−1	−1
1	−1	1	−1
1	1	1	−1
1	1	1	1

$F = 2$

Clearly, the expected value satisfies $E(F) \leq 2$, and by linearity we get

Bell's inequality

$$E(WY) + E(XY) + E(XZ) - E(WZ) \leq 2 \tag{5}$$

Here, from a classical point of view, we may assume

- **reality:** the values P_W, P_X, P_Y, P_Z are intrinsic to the particles, Alice and Bob just discover them,
- **locality:** Alice measurement is independent of Bob's, and conversely.

Let us assume that in the above thought experiment Charly prepares both particles in the entangled state $\mathbf{b}_3 = \frac{1}{\sqrt{2}}(\mathbf{e}_{01} - \mathbf{e}_{10})$ and that the observables are given as

$$\begin{aligned} W &= \sigma_3 & Y &= \frac{1}{\sqrt{2}}(\sigma_1 + \sigma_3) \\ X &= \sigma_1 & Z &= \frac{1}{\sqrt{2}}(\sigma_1 - \sigma_3) \end{aligned} \tag{6}$$

Then,

$$\frac{1}{\sqrt{2}} = E(WY) = E(XY) = E(XZ) = -E(WZ) \tag{7}$$

and $E(F) = \frac{4}{\sqrt{2}} = 2\sqrt{2}$, which contradicts (5).
In order to avoid the paradox, neither realisticity nor locality can be assumed.

Superdense encoding

Bell basis

In $\mathbb{H}_2 = \mathbb{H}_1 \otimes \mathbb{H}_1$, let $\{\mathbf{e}_{00}, \mathbf{e}_{01}, \mathbf{e}_{10}, \mathbf{e}_{11}\}$ be the canonical basis and let

$$\begin{aligned} \mathbf{b}_0 &= \tfrac{1}{\sqrt{2}}(\mathbf{e}_{00} + \mathbf{e}_{11}) & \mathbf{b}_1 &= \tfrac{1}{\sqrt{2}}(\mathbf{e}_{00} - \mathbf{e}_{11}) \\ \mathbf{b}_2 &= \tfrac{1}{\sqrt{2}}(\mathbf{e}_{01} + \mathbf{e}_{10}) & \mathbf{b}_3 &= \tfrac{1}{\sqrt{2}}(\mathbf{e}_{01} - \mathbf{e}_{10}) \end{aligned} \tag{8}$$

The collection $B = \{\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is an ON basis of $\mathbb{H}_2$, it is the so called Bell basis.

Alice wants to transmit to Bob a pair of classical bits, say $\varepsilon = \varepsilon_1 \varepsilon_0$, by transmitting just one qubit.

They agree initially in the entangled state $\mathbf{x} = \mathbf{b}_0$.

1. Alice calculates $\mathbf{y} = \mathbf{b}_\varepsilon$ as follows:

$$\begin{aligned}
\varepsilon = 00 &\implies \mathbf{y} = (\mathbf{1}_1 \otimes \mathbf{1}_1)(\mathbf{x}) = \mathbf{b}_0 \\
\varepsilon = 01 &\implies \mathbf{y} = (\sigma_3 \otimes \mathbf{1}_1)(\mathbf{x}) = \mathbf{b}_1 \\
\varepsilon = 10 &\implies \mathbf{y} = (\sigma_1 \otimes \mathbf{1}_1)(\mathbf{x}) = \mathbf{b}_2 \\
\varepsilon = 11 &\implies \mathbf{y} = ((i\,\sigma_2) \otimes \mathbf{1}_1)(\mathbf{x}) = \mathbf{b}_3
\end{aligned}$$

2. Alice sends $\mathbf{y}$ to Bob.
3. Bob measures $\mathbf{y}$ with respect to the Bell basis,
4. and he recovers $\varepsilon = \varepsilon_1 \varepsilon_0$.

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Space of two qubits

In $\mathbb{H}_1 = \mathbb{C}^2$, let us consider two ON basis $\{\mathbf{e}_0^0, \mathbf{e}_1^0\}$ and $\{\mathbf{e}_0^1, \mathbf{e}_1^1\}$ and the quregister $\mathbf{x} \in \mathbb{H}_2$ of two qubits,

$$\mathbf{x} = x_{00}\mathbf{e}_0^0 \otimes \mathbf{e}_0^1 + x_{11}\mathbf{e}_1^0 \otimes \mathbf{e}_1^1, \text{ with } x_{00}, x_{11} \in \mathbb{C} \;\&\; |x_{00}|^2 + |x_{11}|^2 = 1.$$

If a measurement of the first qubit of $\mathbf{x}$ is performed, with respect to the first basis $\{\mathbf{e}_0^0, \mathbf{e}_1^0\}$, then with a probability $|x_{00}|^2$ its current state will be $\mathbf{e}_0^0$ and the register will transit into $\mathbf{e}_0^0 \otimes \mathbf{e}_0^1$. Similarly, with a probability $|x_{11}|^2$ the current state of the first qubit is $\mathbf{e}_1^0$ and the register will transit into $\mathbf{e}_1^0 \otimes \mathbf{e}_1^1$. Thus, if $\mathbf{e}_i^0$ is assumed by the first qubit then $\mathbf{e}_i^1$ will be assumed by the second qubit. Both outputs are correlated.

An observable over the first qubit acting on 2-quregisters in $\mathbb{H}_2$ is of the form $U \otimes \mathbf{1}_2$, where $U \in \mathbb{C}^{2\times 2}$ is a self-adjoint matrix, and $\mathbf{1}_2$ is the identity matrix of order 2×2. The expected value of the observable is

$$\begin{aligned}
\langle \mathbf{x} | (U \otimes \mathbf{1}_2)\mathbf{x}\rangle &= \langle x_{00}\mathbf{e}_0^0 \otimes \mathbf{e}_0^1 + x_{11}\mathbf{e}_1^0 \otimes \mathbf{e}_1^1 | \\
&\quad (U \otimes \mathbf{1}_2)(x_{00}\mathbf{e}_0^0 \otimes \mathbf{e}_0^1 + x_{11}\mathbf{e}_1^0 \otimes \mathbf{e}_1^1)\rangle \\
&= \langle x_{00}\mathbf{e}_0^0 \otimes \mathbf{e}_0^1 + x_{11}\mathbf{e}_1^0 \otimes \mathbf{e}_1^1 | x_{00} U\mathbf{e}_0^0 \otimes \mathbf{e}_0^1 + x_{11} U\mathbf{e}_1^0 \otimes \mathbf{e}_1^1\rangle \\
&= \quad |x_{00}|^2 \langle \mathbf{e}_0^0 \otimes \mathbf{e}_0^1 | U\mathbf{e}_0^0 \otimes \mathbf{e}_0^1\rangle + \overline{x_{00}} x_{11} \langle \mathbf{e}_0^0 \otimes \mathbf{e}_0^1 | U\mathbf{e}_1^0 \otimes \mathbf{e}_1^1\rangle \\
&\quad + \overline{x_{11}} x_{00} \langle \mathbf{e}_1^0 \otimes \mathbf{e}_1^1 | U\mathbf{e}_0^0 \otimes \mathbf{e}_0^1\rangle + |x_{11}|^2 \langle \mathbf{e}_1^0 \otimes \mathbf{e}_1^1 | U\mathbf{e}_1^0 \otimes \mathbf{e}_1^1\rangle \\
&= |x_{00}|^2 \langle \mathbf{e}_0^0 | U\mathbf{e}_0^0\rangle + |x_{11}|^2 \langle \mathbf{e}_1^0 | U\mathbf{e}_1^0\rangle \qquad (9)
\end{aligned}$$

since, being $\{\mathbf{e}_0^1, \mathbf{e}_1^1\}$ ON, for all $\mathbf{z}_0, \mathbf{z}_1$, $\langle \mathbf{z}_0 \otimes \mathbf{e}_i^1 | \mathbf{z}_1 \otimes \mathbf{e}_j^1\rangle = \langle \mathbf{z}_0 | \mathbf{z}_1\rangle \delta_{ij}$ where δ_{ij} is Kroenecker's delta.

Then

$$\langle \mathbf{x} | (U \otimes \mathbf{1}_2)\mathbf{x} \rangle = \mathrm{tr}(U\rho_{\mathbf{x}}) \ \text{ where } \ \rho_{\mathbf{x}} = \begin{bmatrix} |x_{00}|^2 & 0 \\ 0 & |x_{11}|^2 \end{bmatrix}. \tag{10}$$

The map $\rho_{\mathbf{x}}$ is the density of $\mathbf{x}$, it is positive, self-adjoint, and its trace is 1. Eq. (9) is valid for any observable, in particular for the orthogonal projection $U = \pi_L$, where L is the eigenspace corresponding to an eigenvalue λ of an observable V. From eq's. (9) and (1),

$$\begin{aligned} \langle \mathbf{x} | (\pi_L \otimes \mathbf{1}_2)\mathbf{x} \rangle &= |x_{00}|^2 \langle \mathbf{e}_0^0 | \pi_L \mathbf{e}_0^0 \rangle + |x_{11}|^2 \langle \mathbf{e}_1^0 | \pi_L \mathbf{e}_1^0 \rangle \\ &= |x_{00}|^2 \Pr(\lambda | \mathbf{e}_0^0) + |x_{11}|^2 \Pr(\lambda | \mathbf{e}_1^0), \end{aligned} \tag{11}$$

where $\Pr(\lambda | \mathbf{e}_j^0)$ is the probability to output eigenvalue λ at a measurement in the state $\mathbf{e}_j^0$, $j = 0, 1$.

Eq. (11) may be written as

$$\Pr(\lambda) = p_{00}\langle \mathbf{e}_0^0 | \pi_L \mathbf{e}_0^0 \rangle + p_{10}\langle \mathbf{e}_1^0 | \pi_L \mathbf{e}_1^0 \rangle, \tag{12}$$

where $p_{j0} = |x_{jj}|^2$ is the probability that the measurement is performed at state $\mathbf{e}_j^0$, $j = 0, 1$. Each possible result in the observable V, from the quregister $\mathbf{x}$, occurs with a probability which is a linear combination of the probabilities to get that result from each of the states $\mathbf{e}_j^0$, $j = 0, 1$. The operator $\rho_{\mathbf{x}}$ is realized as an ensemble of the states $\mathbf{e}_j^0$, $j = 0, 1$: the probability of being in any of the $\mathbf{e}_j^0$ is $p_{j0} = |x_{jj}|^2$.

Multidimensional quregisters

Similarly, for n-registers in the space $\mathbb{H}_n$, with basis $\left(\mathbf{e}_i^{(n)}\right)_{i=0}^{2^n-1}$, any $2n$-register is expressed as $\mathbf{x} = \sum_{0\leq i,j\leq 2^n-1} a_{ij}\mathbf{e}_i^{(n)} \otimes \mathbf{e}_j^{(n)}$, with $\sum_{0\leq i,j\leq 2^n-1} |a_{ij}|^2 = 1$. An observable at the first n-register is of the form $U_{2^n} \otimes \mathbf{1}_{2^n}$ and its expected value is

$$\begin{aligned}
\langle \mathbf{x} | (U_{2^n} \otimes \mathbf{1}_{2^n})\mathbf{x}\rangle &= \sum_{0\leq i_0,i_1,j\leq 2^n-1} \overline{a_{i_0 j}} a_{i_1 j} \langle \mathbf{e}_{i_0}^{(n)} | U_{2^n} \mathbf{e}_{i_1}^{(n)}\rangle \\
&= \sum_{j=0}^{2^n-1} \left[\sum_{0\leq i_0,i_1\leq 2^n-1} \overline{a_{i_0 j}} a_{i_1 j} \langle \mathbf{e}_{i_0}^{(n)} | U_{2^n} \mathbf{e}_{i_1}^{(n)}\rangle \right] \\
&= \mathrm{tr}(U_{2^n} \rho_{\mathbf{x}}^{(n)}).
\end{aligned}$$

where

$$\rho_{\mathbf{x}}^{(n)} = \mathrm{tr}\left(\mathbf{x} \cdot \mathbf{x}^T\right) = \sum_{0\leq i_0,i_1,j\leq 2^n-1} \overline{a_{i_0 j}} a_{i_1 j} \left(\mathbf{e}_{i_0}^{(n)}\right) \cdot \left(\mathbf{e}_{i_1}^{(n)}\right)^T = \left[\sum_{j=0}^{2^n-1} \overline{a_{i_0 j}} a_{i_1 j} \right]_{0\leq i_0,i_1\leq 2^n-1}.$$

The operator $\rho_{\mathbf{x}}^{(n)}$ is the density of $\mathbf{x}$. It can be seen that $\rho_{\mathbf{x}}^{(n)}$ is positive, self-adjoint, and its trace is 1.
Consequently, $\rho_{\mathbf{x}}^{(n)}$ is similar to a diagonal matrix whose entries are its eigenvalues, they are indeed real and positive, and sum up to 1.
If $(\mathbf{f}_i)_{i=0}^{2^n-1}$ is a basis representing $\rho_{\mathbf{x}}^{(n)}$ by a diagonal matrix, $\rho_{\mathbf{x}}^{(n)} = \sum_{i=0}^{2^n-1} f_i \left[\mathbf{f}_i \cdot \mathbf{f}_i^T\right]$, with $0 \leq f_i \leq 1$ and $\sum_{i=0}^{2^n-1} f_i = 1$, then $\rho_{\mathbf{x}}^{(n)}$ can be considered as an ensemble of the quregisters $\{\mathbf{f}_i\}_{i=0}^{2^n-1}$.
If just one of the values f_i has absolute value 1 and the others are cero, the ensemble is called pure, otherwise it is mixed. The ensemble is puro if and only if $\left(\rho_{\mathbf{x}}^{(n)}\right)^2 = \rho_{\mathbf{x}}^{(n)}$, and it is mixed if and only if $\left(\rho_{\mathbf{x}}^{(n)}\right)^2 \neq \rho_{\mathbf{x}}^{(n)}$.