The Kochen-Specker Theorem

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1. Observables and Uncertainty
2. C^*-algebra of observables
3. Quantum Logic
4. Gleason's Theorem
5. Kochen-Specker Theorem
   - Theorem's statement
   - Geometrical motivations
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Observables and Uncertainty

$\mathbb{H}$: finite dimensional Hilbert space over $\mathbb{C}$ and
$S_{\mathbb{H}}$: its unit sphere.
A linear map $U : \mathbb{H} \to \mathbb{H}$ is self-adjoint if $\forall \mathbf{x}, \mathbf{y} \in \mathbb{H}$ $\langle \mathbf{x} | U\mathbf{y} \rangle = \langle U\mathbf{x} | \mathbf{y} \rangle$, or equivalently: $U^H = U$.
A self-adjoint map is also called an observable.
For an observable $U : \mathbb{H} \to \mathbb{H}$ there exists an ON basis of $\mathbb{H}$ consisting of eigenvectors of U.
If $\lambda_0, \ldots, \lambda_{k-1}$ are the eigenvalues of U and $L_0, \ldots, L_k$ are the corresponding eigenspaces the following implication holds:

$$\mathbf{x} \in L_\kappa \implies U(\mathbf{x}) = \lambda_\kappa \mathbf{x}.$$

Consequently, U is represented as

$$U = \sum_{\kappa=0}^{k-1} \lambda_\kappa \pi_{L_\kappa}.$$

Since π_{L_κ} is an orthogonal projection, for each $\mathbf{x} \in \mathbb{H}$, $\langle \mathbf{x} - \pi_{L_\kappa}(\mathbf{x}) | \pi_{L_\kappa}(\mathbf{x}) \rangle = 0$, thus

$$\langle \mathbf{x} | \pi_{L_\kappa}(\mathbf{x}) \rangle = \langle \pi_{L_\kappa}(\mathbf{x}) | \pi_{L_\kappa}(\mathbf{x}) \rangle = \| \pi_{L_\kappa}(\mathbf{x}) \|^2.$$

Extended measurement principle

For any observable U, when measuring an n-register $\mathbf{x} \in \mathbb{H}$, the output is an eigenvalue λ_κ and the current state will be the normalized projection $\frac{\pi_{L_\kappa}(\mathbf{x})}{\|\pi_{L_\kappa}(\mathbf{x})\|}$. For each eigenvalue λ_κ, the probability that it is the output is

$$\Pr(\lambda_\kappa) = \langle \mathbf{x} | \pi_{L_\kappa}(\mathbf{x}) \rangle. \tag{1}$$

Evidently, $\sum_{\kappa=0}^{k-1} \Pr(\lambda_\kappa) = \sum_{\kappa=0}^{k-1} \|\pi_{L_\kappa}(\mathbf{x})\|^2 = \|\mathbf{x}\|^2 = 1$.

For any observable U, let $(\mathbf{v}_j)_j$ be an ON basis of $\mathbb{H}$ consisting of eigenvectors of U: λ_j is the corresponding eigenvalue to $\mathbf{v}_j$.
Any $\mathbf{z} \in S_{\mathbb{H}}$, $\mathbf{z} = \sum_i a_i \mathbf{v}_i$, with $\sum_i |a_i|^2 = 1$. And

$$\begin{aligned} \langle \mathbf{z} | U\mathbf{z} \rangle &= \left\langle \sum_i a_i \mathbf{v}_i \middle| U \left(\sum_j a_j \mathbf{v}_j \right) \right\rangle \\ &= \left\langle \sum_i a_i \mathbf{v}_i \middle| \sum_j a_j \lambda_j \mathbf{v}_j \right\rangle \\ &= \sum_i \lambda_i |a_i|^2 = E(\lambda_i) \end{aligned}$$

$\langle \mathbf{z} | U\mathbf{z} \rangle$ is the expected observed value of $\mathbf{z}$ under U.
Standard deviation of U:

$$\triangle U : \mathbb{H} \to \mathbb{R}\ ,\ \mathbf{x} \mapsto \triangle U(\mathbf{x}) = \sqrt{\langle U^2\mathbf{x} | \mathbf{x} \rangle - \langle U\mathbf{x} | \mathbf{x} \rangle^2}.$$

Let $U_1, U_2 : \mathbb{H} \to \mathbb{H}$ be two observables. Then $\forall \mathbf{x} \in \mathbb{H}$:

$$\langle U_2 \circ U_1 \mathbf{x} | \mathbf{x} \rangle \langle \mathbf{x} | U_2 \circ U_1 \mathbf{x} \rangle = |\langle U_1 \mathbf{x} | U_2 \mathbf{x} \rangle|^2 = \langle U_1 \circ U_2 \mathbf{x} | \mathbf{x} \rangle \langle \mathbf{x} | U_1 \circ U_2 \mathbf{x} \rangle,$$

and, from the Schwartz inequality, $|\langle U_1 \mathbf{x} | U_2 \mathbf{x} \rangle|^2 \leq \| U_1 \mathbf{x} \|^2 \| U_2 \mathbf{x} \|^2$.

Robertson-Schrödinger Inequality

$$\frac{1}{4} |\langle (U_1 \circ U_2 - U_2 \circ U_1) \mathbf{x} | \mathbf{x} \rangle|^2 \leq \| U_1 \mathbf{x} \|^2 \| U_2 \mathbf{x} \|^2.$$

The commutator of the two observables is $[U_1, U_2] = U_1 \circ U_2 - U_2 \circ U_1$.
Observables U_1, U_2 are compatible if $[U_1, U_2] = 0$.

Heisenberg Principle of Uncertainty

$$|\triangle U_1(\mathbf{z})|^2 |\triangle U_2(\mathbf{z})|^2 \geq \frac{1}{4} |\langle \mathbf{z} | \, [U_1, U_2] \, \mathbf{z} \rangle|^2 .$$

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C^*-algebra of observables

Quantum system Σ: a collection of observables defined over a complex finite dimensional Hilbert space $\mathbb{H}$,

$\mathbb{A}_\Sigma = \mathcal{L}(\Sigma)$: the C^*-algebra generated by Σ.

A linear functional $f : \mathbb{A}_\Sigma \to \mathbb{C}$ is positive if

$$\forall U \in \mathbb{A}_\Sigma : \ \langle f | U^* U \rangle \geq 0.$$

The identity map $\mathbf{1}$ is an unit in the C^*-algebra $\mathbb{A}_\Sigma$.

Each positive linear functional $f : \mathbb{A}_\Sigma \to \mathbb{C}$ has norm $\|f\| = \langle f | \mathbf{1} \rangle$.

A state over $\mathbb{A}_\Sigma$ is a normalized positive linear functional:

$$f_0, f_1 \text{ states} \quad \Longrightarrow \quad \forall t \in [0,1] : \ (1-t) f_0 + t f_1 \text{ state}.$$

Each point $\mathbf{x}$ in the unit sphere $S_{\mathbb{H}}$ can be identified as a state

$$\mathbf{x} : U \mapsto \langle \mathbf{x} | U \mathbf{x} \rangle \ : \text{ the expected value of } \mathbf{x} \text{ under } U.$$

After Banach-Alaoglu Theorem: In the weak* topology, the unit ball is compact: the set of states is compact in the weak* topology.
For any $U \in \mathbb{A}_\Sigma$, the expected value of U at state $z \in \mathbb{A}_\Sigma^*$ is $E_z(U) = \langle z|U\rangle$.
Spectrum of U: $\Lambda(U) = \{\lambda \in \mathbb{C} | U - \mathbf{1} \text{ is not invertible in } \mathbb{A}_\Sigma\}$.
The uncertainty is the variance of U at z:

$$\mathrm{Var}_z(U) = E_z\left(U - E_z(U)\mathbf{1}\right)^2 = E_z\left(U^2\right) - E_z(U)^2 \geq 0.$$

Heisenberg Principle of Uncertainty is stated as follows:

$$\mathrm{Var}_z(A)\mathrm{Var}_z(B) \geq \frac{1}{2}\left|AB - BA\right|.$$

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Proposition: any observable with eigenvalues $0, 1$, i.e. "yes" or "no" measures. Each proposition is an idempotent self-adjoint operator, $A^2 = A$. Thus they are projections: each proposition corresponds to a subspace in the Hilbert space $\mathbb{H}$.

The "tautological" value 1 corresponds to the whole space $\mathbb{H}$, "inconsistent" value to the null space $\{\mathbf{0}\}$, the "conjunction" to intersection, or "meet", and "disjunction" to "direct sum" or "linear union". "Negation" is thus orthogonal complementarity.

If $A_1, A_2 \in \mathbb{A}_\Sigma$ are propositions then

$$\begin{aligned} \neg A_1 &= \mathbf{1} - A \\ A_1 \wedge A_2 &= \lim_{n \to +\infty} (A_1 A_2)^n \\ A_1 \vee A_2 &= \neg(\neg A_1 \wedge \neg A_2) = \mathbf{1} - \lim_{n \to +\infty} ((\mathbf{1} - A_1)(\mathbf{1} - A_2))^n \end{aligned}$$

Spectral Theorem

Every self-adjoint operator can be split as the direct sum of the orthogonal projections over its eigenspaces.

Every observable is the linear union of propositions that are mutually compatible and compatible with the given observable

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Let $\mathbb{H}$ be a (real, complex, quaternion) Hilbert space and let

$$\mathcal{V}(\mathbb{H}) = \{L < \mathbb{H} |\, L \text{ is a closed linear space in } \mathbb{H}\}.$$

$\mathcal{V}(\mathbb{H})$ is a complete orthomodular lattice.
If $\{L_k\}_{k\in K} \subset \mathcal{V}(\mathbb{H})$ is a collection of subspaces, $\bigoplus_{k\in K} L_\kappa$ is its supremum.
If $\{\mathbf{v}_\iota\}_{\iota\in I} \subset \mathbb{H}$ are vectors, $\mathcal{L}\{\mathbf{v}_\iota\}_{\iota\in I} \in \mathcal{V}(\mathbb{H})$ is the span of the vectors.
A measure is a map $m : \mathcal{V}(\mathbb{H}) \to \mathbb{R}$ such that

$$\begin{array}{rcl} m(\mathbb{H}) & = & 1 \\ \{L_k\}_{k\in K} \text{ pairwise orthogonal } \Longrightarrow \quad m\left(\bigoplus_{k\in K} L_\kappa\right) & = & \sum_{k\in K} m\left(L_\kappa\right) \end{array}$$

For instance, for any $\mathbf{x} \in S_{\mathbb{H}}$, $m_{\mathbf{x}} : L \mapsto \langle \mathbf{x} | \pi_L \mathbf{x} \rangle$ is a measure.

Theorem (Gleason, 1957)

Let $\mathbb{H}$ be a separable Hilbert space of dimension at least 3. For each measure m there exists a Hermitian positive operator T_m such that

$$\forall L \in \mathcal{V}(\mathbb{H}) : \quad m(L) = \mathrm{Tr}(T_m \pi_L).$$

Besides, the map $m \leftrightarrow T_m$ is a bijection among measures and Hermitian positive operators.

Since $S_{\mathbb{H}}$ is connected and the map $L \mapsto \mathrm{Tr}(T_m \pi_L)$ is continuous (in a topology well defined in $\mathcal{V}(\mathbb{H})$) then, from Gleason's Theorem, there are no ("yes","no")-measures in the space of propositions.
Since $S_{\mathbb{H}}$ is weak*-compact, there is finite collection of subspaces in $\mathcal{V}(\mathbb{H})$ in which no non-trivial two-valued measure can be defined. A construction of such an example is given by Kochen-Specker Theorem.

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Kochen-Specker Theorem

$\mathbb{H}$: a separable complex Hilbert space. Σ: collection of observables.
$\mu : \Sigma \to \mathbb{R}$: a map assigning to an observable a definite value.

Property (Value definiteness (VD))

At any time μ is a total map: The values of the observables are well determined at any time.

Property

The map μ fulfills the following two rules:

Sum rule *If $U_0, U_1, U_2 \in \Sigma$ are compatible then:*
$$[U_2 = U_0 + U_1 \implies \mu(U_2) = \mu(U_0) + \mu(U_1)].$$

Product rule *If $U_0, U_1, U_2 \in \Sigma$ are compatible then:*
$$[U_2 = U_0 U_1 \implies \mu(U_2) = \mu(U_0)\mu(U_1)].$$

If μ is a "yes"-"no" assignment and $(L_i)_i$ is a finite set of propositions (their projections π_{l_i} have eigenvalues 0,1), the sum rule implies:

$$1 = \mu\left(\pi_{\bigoplus_i L_i}\right) = \sum_i \mu\left(\pi_{L_i}\right).$$

Thus necessarily:

$$\text{(VC)}\quad \left[\mu\left(\pi_{L_i}\right) = 1 \implies \mu\left(\pi_{L_j}\right) = 0\ \forall j \neq i\right].$$

The product rule implies:

$$\text{(VE)}\quad \left[L < L_0 \oplus L_1 \implies \mu\left(\pi_L\right)\mu\left(\pi_{L_0 \oplus L_1}\right) = \mu\left(\pi_L\right)\right].$$

Paint in color red those propositions s.t. $\mu(\pi_{L_i}) = 1$ and in green those propositions s.t. $\mu(\pi_{L_i}) = 0$.

The sum rule and condition **(VC)** imply that in any finite ON system of vectors exactly one vector is red and the others are green.

Condition **(VE)** implies that any proposition in the span of two green propositions should be colored green.

Theorem (Kochen-Specker (KSThm, 1967))

Let $\mathbb{H}$ be a separable complex Hilbert space of dimension at least 3. Then there exists a collection of observables Σ such that **Property 1** *and* **Property 2** *cannot hold simultaneously.*

Theorem (Geometrical tridimensional real Kochen-Specker –)

In $\mathbb{R}^3$ there exists a collection of rays Σ such that the following cannot hold simultaneously for any "green"-"red" coloring:

1. *In any triplet of orthogonal rays exactly one is colored red.*
2. *Any ray lying in the span of two green rays is colored green.*

Geometrical motivation

$\mathbf{x} = (\cos(\theta)\cos(\phi), \sin(\theta)\cos(\phi), \sin(\phi))$: a point in the unit sphere of $\mathbb{R}^3$.
θ: the longitude. ϕ: the latitude.
$\mathbf{x}_E = (-\sin(\theta), \cos(\theta), 0)$:A point in the equator orthogonal to $\mathbf{x}$ (its longitude is that of $\mathbf{x}$ shifted by an angle of $\pi/2$ radians).
$\mathbf{x}^{\perp} = \mathbf{x} \times \mathbf{x}_E = (-\cos(\theta)\sin(\phi), -\sin(\theta)\sin(\phi), \cos(\phi))$.
$\{\mathbf{x}, \mathbf{x}_E, \mathbf{x}^{\perp}\}$ is a positively oriented ON basis of $\mathbb{R}^3$.
Let us suppose $\mathbf{x}$ is in the northern hemisphere of the unit sphere and it is not the North Pole. By changing the direction of $\mathbf{x}_E$ if necessary, we may assume that $\mathbf{x}^{\perp}$ lies also in the northern hemisphere.
The circle with center at the origin passing over $\mathbf{x}$ and $\mathbf{x}_E$ is the descent circle of $\mathbf{x}$.

Mapamundi centered at San Sebastián

A descent sequence $\{\mathbf{x}_i\}_{i=0}^{k}$ is a sequence built as follows: $\mathbf{x}_0$ is a point in the northern hemisphere which is not the North Pole. For any $i \geq 0$, $\mathbf{x}_{i+1}$ is a point chosen on the descent circle of $\mathbf{x}_i$ (to the south of $\mathbf{x}_i$).

Lemma

Given two points in the northern hemisphere with different latitude, there is a descent sequence beginning with the northern point and ending in the southern point.

The proof is direct by using the projection $\mathbf{x} \mapsto \frac{1}{x_3}\mathbf{x}$ mapping each point in the northern hemisphere into the crossing of its ray with the plane parallel to the x, y-plane passing by the North Pole. Parallel circles corresponding to same latitudes are mapped into concentric circles, and descent circles into tangents to concentric circles.

Sketch of Thm 3

Let initially $\Sigma = \{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2\}$ be the collection of the three canonical basic vectors. Let us assume that the North Pole $\mathbf{e}_2 = (0, 0, 1)$ is colored red. Let $\mathbf{x}$ be a vector of latitude $\frac{1}{3}\pi$. Then $\mathbf{x}_E$ is in the equator, hence colored green, and $\mathbf{x}^\perp$ has latitude $\frac{1}{6}\pi$. It follows that $\mathbf{x}$ and $\mathbf{x}^\perp$ have opposite colors.

If $\mathbf{x}$ is green then on one side $\mathbf{x}^\perp$ is red and on the other any ray in the descent circle of $\mathbf{x}$ is green. Thus any ray reached by a descent sequence from $\mathbf{x}$ should be green, in particular $\mathbf{x}^\perp$ and this is a contradiction.

Otherwise, any ray at an angle of $\frac{1}{6}\pi$ from the North Pole, shall be red, as is that pole. Indeed any ray in the cone of angle $\frac{1}{6}\pi$ of a red ray is red. Then there is an arc of three red rays from the North Pole to the Equator, another arc of three red rays along the Equator, and another arc of three red rays from the Equator to the North Pole. The three corners in this circuit are red, but they form an orthogonal system. This is a contradiction.

Roughly speaking, KSThm implies that Quantum Mechanics (QM) is not consistent with the following two properties:

Value definiteness (VD) All observables have definite values at all times.

Non Contextuality (NC) If an observable aquires a value, it does so independently of any measurement context.

In symbols:

$$\text{KSThm: } QM \not\models VD + NC$$

Consequently, acceptance of QM entails a rejection of either VD or NC.
VD is the origin of any hidden variables programme.
NC is the motivation of the notion of realism.
It is an important problem how to come up with a version of QM containing VD but not NC.

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